

Stability of Steel Columns with Non-Uniform Cross-Sections

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Abstract: In this work, non-uniform steel members with or without initial geometrical or loading imperfections, that are loaded by axial forces applied concentrically or eccentrically and by concentrated moments applied at the ends or at intermediate points, are studied. More specifically, steel members with varying cross-sections, tapered or stepped or members consisting by two different tapered parts are considered. The formulation presented in this work is based on solving the governing equation of the problem through a numerical method where the eigenshapes of the member are employed. A failure plasticity criterion is introduced for members especially the short ones that will never reach the elastic critical buckling load. Although only the simply supported beam-column case is studied herein, it is obvious that the method can be extended to multi-span beams and frames, by employing the corresponding eigenshapes. Useful diagrams are presented for both the critical buckling loads and the equilibrium paths showing the influence of the main characteristics of the beam-column.

Keywords: Steel columns, stability, non-uniform members, buckling loads, imperfections, yield failure.

INTRODUCTION

The use of steel members with non-uniform cross-sections either as columns or as distressed parts of a structure with or without bending moments is very common in steel constructions. There is a wide variety of structures such as buildings frames, bridge members, masts or cranes, etc, which are designed with members of non-uniform cross-sections in order to minimize the required material. The significance of using such members in structures and the necessity to study their structural behavior has been realized since the beginning of the 19th century by A. Föppl, as it was referenced by Timoshenko [1].

A first approach and study of the above-mentioned problems of columns with variable cross-sections was made by Dinkik in 1914 and in 1916. The main results of these studies were translated in English by Malets [2, 3]. The same problem was studied by Ostwald [4], by Ono [5], by Stedman [6] and by Morley [7, 8]. On the history of early studies on these topics, one can refer to Timoshenko [9].

Bleich [10] studied compression members the cross-sectional moment of inertia of which was varying by a half-sine curve. On the other hand, the significance of the initial imperfections was noted very early and studied mainly experimentally by Marston [11], by Jensen [12] and by Lilly [13], the studies of which were gathered by Salmon [14]. A significant study of compression stepped columns consisting by two parts through the use of the Galerkin method has been presented by Dimitrof [15].

There is a relatively large number of theoretical or experimental publications on tapered or stepped columns with or without imperfections [16-31]. In the present paper,

non-uniform steel members with or without imperfections (of any form), loaded by axial forces (concentrically or eccentrically applied) and by concentrated moments applied at its ends or intermediate points are studied. The steel members with cross-section that may vary along the length, can be tapered or stepped or they can be members consisting by two unequal tapered parts. The imperfections considered may have any form. The formulation presented in this paper is based on solving the governing equation of the problem through the Galerkin method using the eigenshapes of the member. A plasticity failure criterion is introduced for stub or short members that will never reach the elastic critical buckling load. Although in this paper only the simply supported single-span beam-column is studied, it is obvious that the formulation presented may be extended to any type of frame members or frames using the corresponding eigenshapes.

The results are presented in the form of diagrams either for the critical buckling loads or for the equilibrium paths, showing the influence of the main member's characteristics, as for example the cross-sectional variation law, the intermediate loads and bending moments or the existing imperfections on the above mentioned buckling loads and equilibrium paths.

Although these diagrams are derived for a simply supported beam-column they can be readily employed for the design of steel frames with such members through the use of the equivalent buckling length factor concept.

GOVERNING EQUATIONS

Let us consider the beam-column shown in Fig. (1), where the cross-section varies along the length (x -axis) according to a known shape (parabolic, tapered, stepped etc). The considered member is loaded by forces and moments as shown in Fig. (1). Moreover, the member may have an initial imperfection $w_0(x)$.

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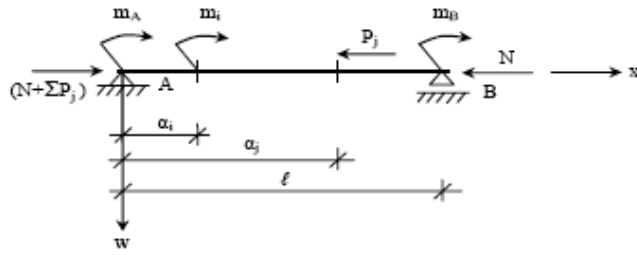


Fig. (1). Forces and moments on a beam-column.

In addition, we assume that the member is laterally supported and hence, it is protected against the possibility of buckling about the weak axis. The differential equation for buckling of the beam-column is given as follows:

$$[EI(x)w'''] + N(w + w_0)'' + \sum_j P_j(w + w_0)''[1 - H(x - \alpha_j)] - \sum_i m_i \delta'(x - \alpha_i) = 0 \tag{1}$$

where EI is the bending rigidity of the member, w is the deflected shape, w₀ is the initial imperfection, N is the externally applied axial force, P_i and m_i are concentrated forces and moments applied at intermediate positions over the length, H(x) is the Heaviside unit step function and δ(x) the Dirac delta function.

The above equation may be written as follows:

$$EIw'''' + 2EI'w'''' + EI''w'''' + Nw'''' + \sum_j P_j w'''' [1 - H(x - \alpha_j)] = -Nw_0'''' - \sum_j P_j w_0'''' [1 - H(x - \alpha_j)] + \sum_i m_i \delta'(x - \alpha_i) \tag{2}$$

Non-existence of moments and initial imperfection leads to a homogeneous differential equation (without second member), which gives the buckling critical loads, while existence of moments and initial imperfections leads to a complete differential equation, which allows us to study the equilibrium paths.

We are searching for a solution in the following form:

$$w(x) = c_1 X_1(x) + c_2 X_2(x) + \dots + c_n X_n(x) \tag{3}$$

where X_n(x) = sin(nπx / l) are the eigenshapes of the simply supported beam and c_n are constants to be determined.

Introducing eq(3) into eq(2) we obtain:

$$EI \sum_\rho c_\rho \left(\frac{\rho\pi}{\ell}\right)^4 \sin \frac{\rho\pi x}{\ell} - 2EI' \sum_\rho c_\rho \left(\frac{\rho\pi}{\ell}\right)^3 \cos \frac{\rho\pi x}{\ell} - EI'' \sum_\rho c_\rho \left(\frac{\rho\pi}{\ell}\right)^2 \sin \frac{\rho\pi x}{\ell} - N \sum_\rho c_\rho \left(\frac{\rho\pi}{\ell}\right) \sin \frac{\rho\pi x}{\ell} - \sum_j \left\{ \frac{N}{\mu_j} \sum_\rho c_\rho \left(\frac{\rho\pi}{\ell}\right)^2 \sin \frac{\rho\pi x}{\ell} [1 - H(x - \alpha_j)] \right\} - Nw_0'' - \sum_j \frac{N}{\mu_j} w_0'' [1 - H(x - \alpha_j)] + \sum_i m_i \delta'(x - \alpha_i) \tag{4a}$$

where we have set:

$$P_j = N / \mu_j \tag{4b}$$

where μ_i are the ratios of the loads P_i over N.

BUCKLING LOADS

By setting w₀=0 and m_i=0, eq(4a) becomes:

$$EI \sum_\rho c_\rho \left(\frac{\rho\pi}{\ell}\right)^4 \sin \frac{\rho\pi x}{\ell} - 2EI' \sum_\rho c_\rho \left(\frac{\rho\pi}{\ell}\right)^3 \cos \frac{\rho\pi x}{\ell} - EI'' \sum_\rho c_\rho \left(\frac{\rho\pi}{\ell}\right)^2 \sin \frac{\rho\pi x}{\ell} - N \sum_\rho c_\rho \left(\frac{\rho\pi}{\ell}\right) \sin \frac{\rho\pi x}{\ell} - \sum_j \frac{N}{\mu_j} \sum_\rho c_\rho \left(\frac{\rho\pi}{\ell}\right)^2 \sin \frac{\rho\pi x}{\ell} [1 - H(x - \alpha_j)] = 0 \tag{5}$$

Applying the Galerkin procedure in eq(5) and taking into account the orthogonality condition we obtain:

$$c_1(A_{k1} - \alpha_{k1}N) + c_2(A_{k2} - \alpha_{k2}N) + \dots + c_n(A_{kn} - \alpha_{kn}N) = 0 \quad \text{with } k = 1 \text{ to } n \tag{6a}$$

where:

$$A_{k\rho} = E \int_0^\ell I(x) \left(\frac{\rho\pi}{\ell}\right)^4 \sin \frac{\rho\pi x}{\ell} \sin \frac{k\pi x}{\ell} dx - 2E \int_0^\ell I'(x) \left(\frac{\rho\pi}{\ell}\right)^3 \cos \frac{\rho\pi x}{\ell} \sin \frac{k\pi x}{\ell} dx - E \int_0^\ell I''(x) \left(\frac{\rho\pi}{\ell}\right)^2 \sin \frac{\rho\pi x}{\ell} \sin \frac{k\pi x}{\ell} dx \tag{6b}$$

$$\alpha_{k\rho} = \frac{\rho^2\pi^2}{2\ell} + \mu_j \int_0^\ell \frac{\rho^2\pi^2}{\ell^2} (\sin \frac{\rho\pi x}{\ell})^2 [1 - H(x - \alpha_j)] dx \quad (\rho = k)$$

$$\alpha_{k\rho} = \frac{\rho^2\pi^2}{2\ell} + \mu_j \int_0^\ell \frac{\rho^2\pi^2}{\ell^2} \sin \frac{\rho\pi x}{\ell} \sin \frac{k\pi x}{\ell} [1 - H(x - \alpha_j)] dx \quad (\rho \neq k)$$

In order that the above linear homogeneous system eq(6a) has non-trivial solutions the determinant of the coefficients of the unknowns c_ρ must be equal to zero. This condition leads to the following eigenvalue problem expressed by the following equation:

$$\|A_{k\rho} - \alpha_{k\rho}N\| = 0 \tag{7}$$

Equation (7) gives the spectrum of the critical buckling loads N_{cr}.

EQUILIBRIUM PATHS

Equation (4) allows us to determine equilibrium paths. Applying once again the Galerkin procedure in eq(4) and taking into account the orthogonality condition we obtain:

$$c_1(A_{k1} - \alpha_{k1}N) + c_2(A_{k2} - \alpha_{k2}N) + \dots + c_n(A_{kn} - \alpha_{kn}N) = B_k \quad \text{with } k = 1 \text{ to } n \tag{8a}$$

where A_{kρ} and α_{kρ} are given in eq(6b), while B_k is given by the following equation:

$$B_k = -N \int_0^\ell w_0'' \sin \frac{k\pi x}{\ell} dx - \sum_j \mu_j N \int_0^{\alpha_j} w_0'' \sin \frac{k\pi x}{\ell} dx - \sum_i m_i \frac{k\pi}{\ell} \cos \frac{k\pi \alpha_i}{\ell} \tag{8b}$$

From the system of equations (8a), we obtain the unknown constants $c_\rho(N)$, ($\rho = 1$ to n) and from eq(3) the equilibrium paths as follows

$$w(x) = c_1(N) \sin \frac{\pi x}{\ell} + \dots + c_n(N) \sin \frac{n \pi x}{\ell} \quad (9)$$

FAILURE CRITERION

It is obvious that there are cases (especially for short or stub columns), where the member will never reach the elastic critical buckling load N_{cr} . Thus, it is necessary to introduce a plasticity criterion, which can be expressed as follows:

$$\frac{N}{A(x)} + \frac{M(x)}{W(x)} \leq \sigma_f \text{ or}$$

$$N + \frac{A(x)}{I(x)} z(x) M(x) \leq \sigma_f A(x) \text{ or}$$

$$N \leq \sigma_f A(x) - \frac{z(x)}{i(x)^2} M(x) = \sigma_f A(x) - \frac{z(x)}{[i(x)/\ell]^2} \cdot \frac{M(x)}{\ell^2} \text{ or,}$$

finally

$$N \leq \sigma_f A(x) - z(x) \lambda^2(x) \frac{M(x)}{\ell^2} \quad (10)$$

where $z(x)$ is the distance of the extreme fiber of the member's cross-section at position x , and $\lambda(x)$ is the slenderness of the member.

INITIAL IMPERFECTIONS

Initial imperfections may appear on a member due to a bad stacking during the transportation of steel members and, rather rarely, due to constructional causes. In any case, initial imperfections have usually a form resembling to a parabolic type that can be expressed as follows:

$$w_o(x) = -4f \left(\frac{x}{\ell} \right)^2 + 4f \frac{x}{\ell} \quad (11)$$

where f is the maximum deformation at $x = \ell/2$.

CROSS-SECTIONAL TYPES

The beam-column cross section may have one of the forms shown in Fig. (2). In Fig. (2a), one can see a column the cross-section of which changes parabolically. In Fig. (2b) a tapered member is shown, while in Fig. (2c) one can see a member composed by two tapered parts with different length. Finally, in Fig. (2d), one can see a stepped beam-column.

In the cases of Fig. (2a,b and c), the cross-section at any point x has usually constant flanges ($b \cdot t_f$) but a web with variable depth z .

In all the above cases, the following relation is valid:

$$I(x) = \frac{4}{3} \cdot t_w \cdot z^3(x) + 2 \cdot b \cdot t_f \cdot \left[z(x) + \frac{t_f}{2} \right]^2 \quad (12)$$

The Parabolic Beam

If the web depth z or the moment of inertia $I(x)$ change parabolically, it will be (see Fig. 2a and 2e):

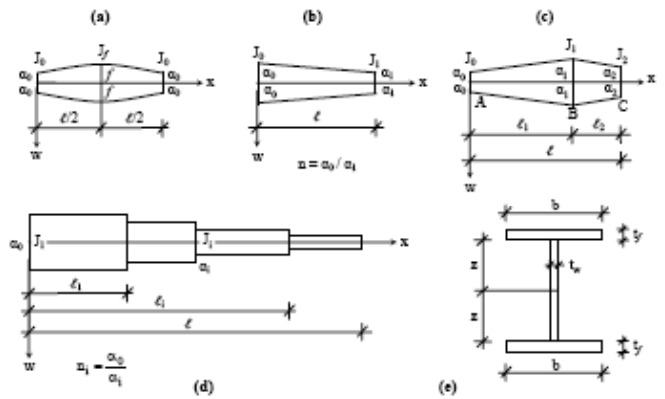


Fig. (2). Steel members with different forms.

$$z(x) = 4(\alpha_o - f) \left(\frac{x}{\ell} \right)^2 - 4(\alpha_o - f) \cdot \frac{x}{\ell} + \alpha_o \quad (13a)$$

$$I(x) = 4(I_o - I_f) \left(\frac{x}{\ell} \right)^2 - 4(I_o - I_f) \cdot \frac{x}{\ell} + I_o \quad (13b)$$

The Tapered Beam

For the case of Fig. (2b), and assuming that $\alpha_1 = n\alpha_o$, we will have:

$$z(x) = \alpha_o(n-1) \cdot \frac{x}{\ell} + \alpha_o \quad (14)$$

The Beam with Two Tapered Parts

For the case of Fig. (2c), and assuming that $\alpha_1 = n_1 \cdot \alpha_o$ and $\alpha_2 = n_2 \cdot \alpha_1$ we will have:

$$z(x) = \alpha_o(n_1-1) \cdot \frac{x}{\ell_1} + \alpha_o \text{ for } 0 \leq x \leq \ell_1 \quad (15a)$$

$$z(x) = n_1 \alpha_o(n_2-1) \cdot \frac{x-\ell_1}{\ell_1} + n_1 \alpha_o \text{ for } \ell_1 \leq x \leq \ell \quad (15b)$$

For members consisting of two tapered parts we note that the discontinuity at $x=\ell_1$, affects the results obtained by using some commercial mathematical manipulators. For the case of a beam composed by two parts with lengths ℓ_1 and ℓ_2 , and flanges AB and BC (Fig. 2c) that have the same inclination, the following approaching formula that removes the aforementioned discontinuity is suggested:

$$z(x) = \alpha_o \left[\sqrt{\left[\frac{n-1}{\ell_1} \cdot (x-\ell_1) \right]^2 + \frac{1}{10\,000}} - \left(n + \frac{1}{10\,000} \right) \right] \quad (15c)$$

with $n = \frac{\alpha_1}{\alpha_o}$

The Stepped Beam-Column

In this case (Fig. 2d) we have:

$$I_1(x) = I_i = \text{const. for } \sum_{\rho=1}^{i-1} \ell_\rho \leq x \leq \sum_{\rho=1}^i \ell_\rho \quad (16)$$

Table 1. IPE Steel Profiles

Profile	IPE 200	IPE 400	IPE 600
2-z (m)	0.183	0.373	0.562
b (m)	0.100	0.180	0.220
t _w (m)	0.0056	0.0086	0.0120
t _f (m)	0.0085	0.0135	0.0190
A (m ²)	0.002735	0.008068	0.01510
I·10 ⁻⁴ (m ⁴)	0.213	2.559	10.604

Table 2. HEB Steel Profiles

Profile	HEB 200	HEB 400	HEB 600
2-z (m)	0.170	0.352	0.540
b (m)	0.200	0.300	0.300
t _w (m)	0.0090	0.0135	0.0155
t _f (m)	0.0150	0.0240	0.0300
A (m ²)	0.00753	0.01915	0.02637
I·10 ⁻⁴ (m ⁴)	0.5871	6.071	18.688

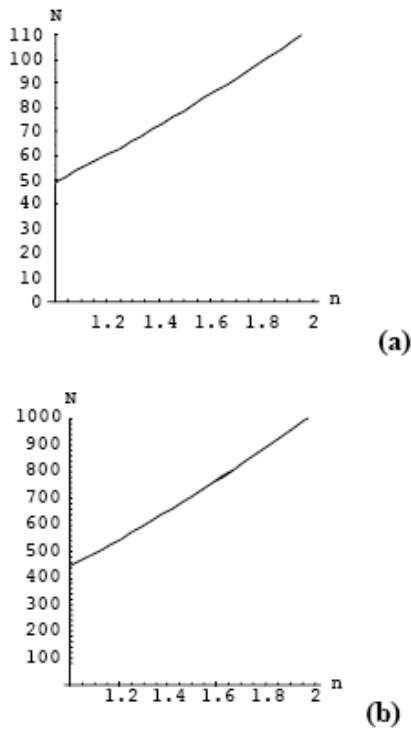


Fig. (3). Convergence study for IPE 200, (a) L=30m and (b) L=10m.

NUMERICAL RESULTS

In order to study the buckling behavior of steel members with non-uniform cross section, we will use members having at x=0 the characteristic properties given by the following Tables 1 and 2 which refer to IPE and HEB standard profiles, respectively, according to European Norms for characterization of standard steel sections.

CONVERGENCE STUDY OF THE METHOD

Studying first the convergence of the method, we see from the plots of Fig. (3) that the results arising from a solution using the first three eigenshapes and another solution using the first six eigenshapes coincide fully. We notice only

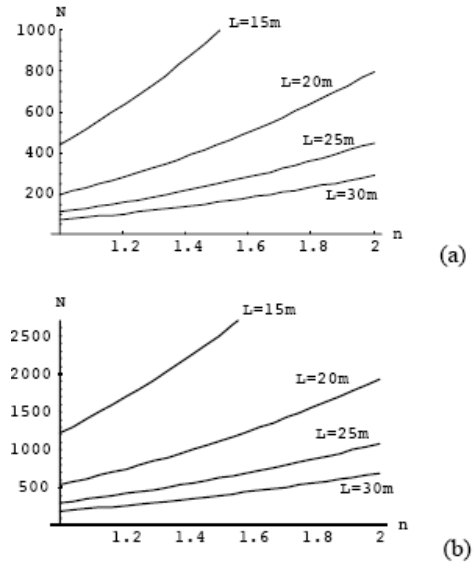


Fig. (4). Critical loads for parabolic beams made from (a) IPE200 and (b) HEB200 profiles with lengths 15, 20, 25 and 30m.

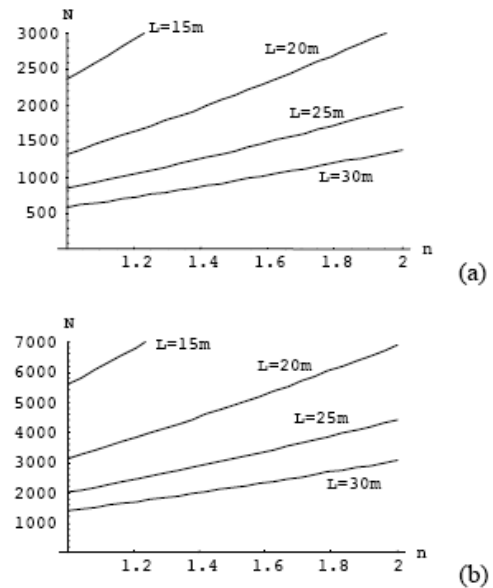


Fig. (5). Critical loads for tapered beams made from (a) IPE400 and (b) HEB400 profiles with lengths 15, 20, 25 and 30m.

a slight mismatch less than 0.1% for values of n ≈ 1.7 and for beam length L=10m. This very small difference is probably due to the numerical approach of the program used.

We see also that for n=1 we recover the Euler critical buckling loads $P_{cr} = \pi^2 EI / L^2$ corresponding to simply supported axially compressed columns.

DETERMINATION OF THE CRITICAL LOAD

Applying the proposed formulae, we determine the critical loads N_{cr} versus n for columns with different lengths and cross-sectional types.

In the following figures, the plots of critical loads are shown for beams with parabolic form made from IPE200 and HEB200 standard profiles (Fig. 4), with tapered form made from IPE400 and HEB400 profiles (Fig. 5), for a beam

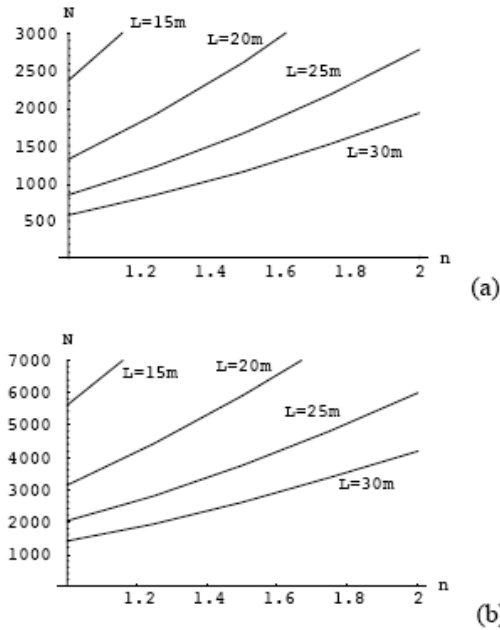


Fig. (6). Critical loads for beams with two tapered pieces made from (a) IPE400 and (b) HEB400 profiles with lengths 15, 20, 25 and 30m.

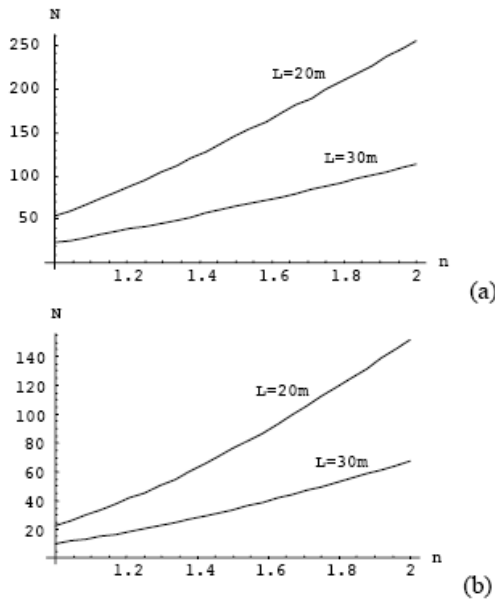


Fig. (7). Critical loads for a stepped beam made from IPE200 with total length 20 and 30m, with (a) $\mu_1=1, \mu_2=1$ and (b) $\mu_1=5, \mu_2=3$.

composed by two tapered parts (Fig. 6), and for a stepped beam (Fig. 7) composed by three equal length parts with equal $n_1=n$, where the part with the smaller cross-section is IPE200 and is loaded by a force N at ℓ , by $P_1=\mu_1N$ at ℓ_1 and by $P_2=\mu_2N$ at ℓ_2 .

In Fig. (5a) it is $\mu_1=0, \mu_2=0$, while in Fig. (5b) it is $\mu_1=5, \mu_2=3$.

From Figs (4 to 7), we can easily see that the critical loads N_{cr} are significantly higher than the ones corresponding to prismatic members (uniform cross-section) depending on the taper ratio n. As the taper ratio n increases the critical load also increases for all types of cross-sectional variation.

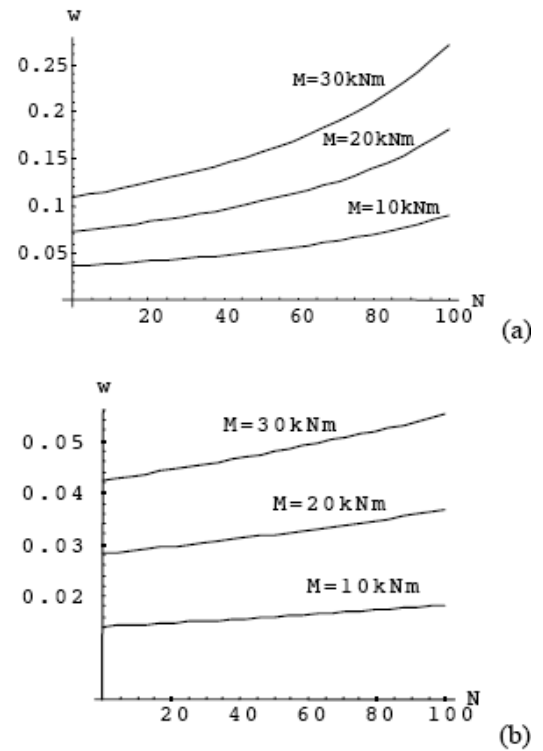


Fig. (8). Equilibrium paths for a parabolic beam made from IPE200 with (a) $n=1.25$ and (b) $n=2.0$.

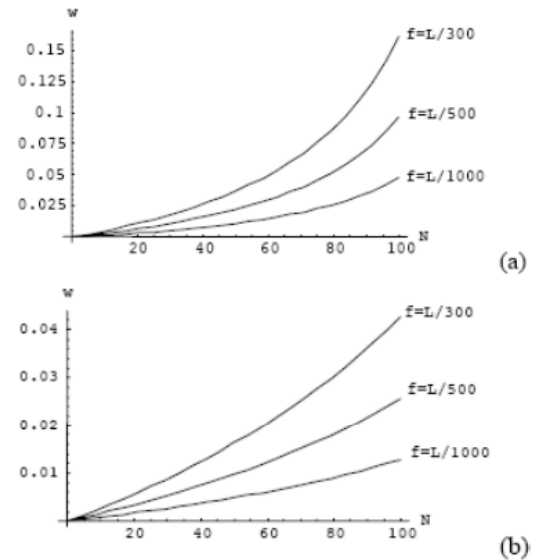


Fig. (9). Equilibrium paths of a tapered beam with (a) $n=1.25$ and (b) $n=2.0$ and initial imperfections f.

Equilibrium Paths

In the plots of Fig. (8), the equilibrium paths versus the axial load N are shown for a beam with parabolic form, length 20m and $n=1.25$ (Fig. 8a) and $n=2$ (Fig. 8b) which is loaded by moments of 10, 20 and 30 kNm.

The Effect of Initial Imperfections

In the plots of Fig. (9), the equilibrium paths versus N of a tapered beam with length 20m and $f=L/300, L/500, L/1000$ with $n=1.25$ (Fig. 9a) or $n=2$ (Fig. 9b) are shown.

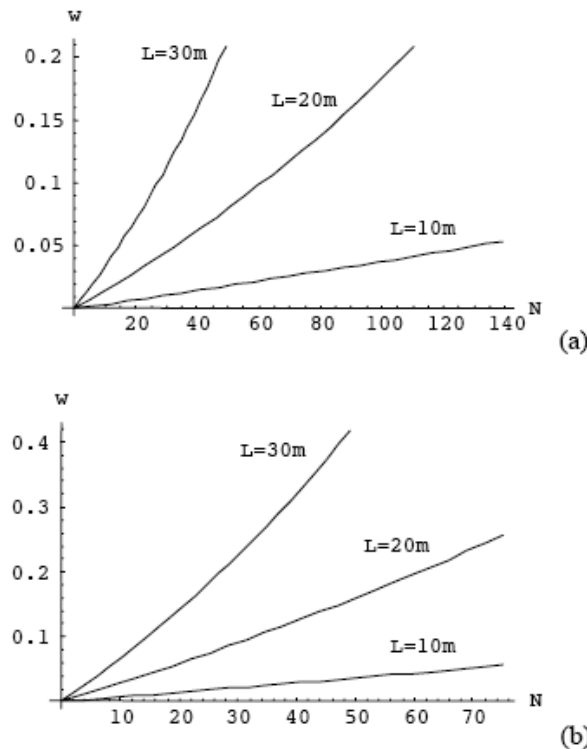


Fig. (10). Equilibrium paths of a parabolic beam loaded eccentrically with (a) $e=0.5\text{m}$ and (b) $e=1.0\text{m}$.

Eccentrically Applied Loads

Finally, the plots of Fig. (10) show the equilibrium paths for the special case of a parabolic beam with length 10, 20, or 30m and $n=2$, which is loaded by an axial force N acting eccentrically at $e=0.5\text{m}$ (Fig. 10a) and $e=1.0\text{m}$ (Fig. 10b).

Some of the above results have been verified via the finite element method. More specifically, a tapered beam with lengths 20m and 30m made from IPE 200 profile and taper ratio $n=1.5$ has been modeled and analyzed for linear buckling using the SAP-2000 v11 commercial FE code. The deviation between analytical and FE results is 2.07% for $L=20\text{m}$ and 1.23% for $L=30\text{m}$ (Fig. 5a). Moreover, a stepped beam with lengths 20m and 30m made from IPE 200 profile and taper ratio $n=1.5$ with $\mu_1=\mu_2=1$ has also been modeled and analyzed with the same code. In this case, the deviation between analytical and FE results is 1.64% for $L=20\text{m}$ and 1.15% for $L=30\text{m}$ (Fig. 7a).

SUMMARY AND CONCLUSIONS

In this paper, a simple and efficient method for the study of non-uniform steel members with or without imperfections, loaded by axial forces concentrically or eccentrically applied and by moments at its ends or at intermediate points is presented. The governing equation of the problem including all the above parameters is solved by the Galerkin method using the eigenshapes of the member.

The variation of the members' cross-section may be whichever as well as the type of initial imperfections. The accuracy of the method is proven excellent using only the three first eigenshapes. A plasticity criterion is applied in order to predict material failure due to buckling deformation.

The existence of bending moments or intermediate forces reduces not only the critical buckling load, but also the load arising from the failure criterion.

The numerical results and the diagrams presented in this study refer to a number of cross-sections made from IPE and HEB standard profiles that are commonly used in steel structures.

The presented formulae can be easily formulated in a personal computer using one of the available commercial programs (such as Maple, Mathematica, Matlab, etc). Although in this work only the simply supported single-span beam-column is presented, it is obvious that the formulation can be extended to any type of frame members or frames using the corresponding eigenshapes.

The results are presented in the form of diagrams either for the critical buckling loads or for the equilibrium paths, showing the influence of the main member's characteristics, as for example the cross-sectional variation law, the intermediate loads and bending moments or the existing imperfections on the above mentioned buckling loads and equilibrium paths. Although these diagrams are derived for a simply supported beam-column they can be readily employed for the design of steel frames with such members through the use of the equivalent buckling length factor concept.

CONFLICT OF INTEREST

None declared.

ACKNOWLEDGEMENT

None declared.

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Received: September 15, 2011

Revised: December 01, 2011

Accepted: December 02, 2011

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